PhD-project:

Measures for analytic functions in several variables.

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Prequisitaries: *If you want to work with this project you should be interested in analysis in general, and have taken basic courses in complex analysis (in one variable), measure theory and functional analysis. Moreover, you should have a solid background in at least one of these subjects.*

Background: The main objects are versions of **Herglotz-Nevanlinna functions**, these are analytic functions $f : \mathcal{D} \to \mathbb{C}^+ \cup \mathbb{R}$, i.e., mapping its domain $\mathcal{D}$ into the closed complex upper halfplane $\mathbb{C}^+$. The case when $\mathcal{D}$ is the upper halfplane $\mathbb{C}^+$ (or equivalently, the unit disc $\mathbb{D}$) is very well studied and such functions are used at very many different places in both pure mathematics and applications (spectral theory, passive systems, homogenization, just to name a few).

In particular, it well known that a function belongs to this class if and only if it can be written as an integral

$$f(z) = a + bz + \int_{\partial\mathcal{D}} K_{\partial\mathcal{D}}(z,t)d\mu(t), \tag{1}$$

with parameters $a \in \mathbb{R}$, $b \geq 0$, μ is a **Borel measure on the boundary of** $\mathcal{D}$, and $K_{\mathcal{D}}$ is a suitable kernel function.

Equivalently, this class can also be characterized via representations with resolvents. In particular, an analytic function $f : \mathbb{C}^+ \to \mathbb{C}$ (with an additional growth condition) satisfies $\mathrm{Im} f(z) \geq 0$ if and only if there is a Hilbert space $\mathcal{H}$, an element $u \in \mathcal{H}$, a **self-adjoint operator** A, and a constant $c \in \mathbb{R}$ such that

$$f(z) = c + \big((A - z)^{-1}u, u\big). \tag{2}$$

Since these representations are essentially unique, there is a close relation between the behaviour of the function f and the properties of the measure μ as well as of the operator A (in particular, its spectral properties). This relation is very well investigated, an overview of classical results can be found e.g., in the book [T].

Recent developments: Instead of functions in one variable, also Herglotz-Nevanlinna functions in several variables have been considered, i.e., analytic functions $f : \mathcal{D} \to \mathbb{C}^+ \cup \mathbb{R}$, where $\mathcal{D} = (\mathbb{C}^+)^n$ is the poly upper halfplane.

It has turned out that also in this case there are representations with some kind of resolvents at least for a large subclass of these functions, see [AMY, ATuY], as well as an integral representation of the form (1) with an appropriate kernel $K_{\partial\mathcal{D}}$ and where $\partial\mathcal{D}$ becomes $\mathbb{R}^n$, the distinguished boundary of $(\mathbb{C}^+)^n$, see [LN1].

However, there is one decisive difference to the one-dimensional case, where each measure (such that the integral exists) defines a function with values in the upper halfplane: The admissible measures in $\mathbb{R}^n$ have to satisfy additional properties, with are quite restrictive, but not yet completely understood.

The project: *The project deals with the relation between Herglotz-Nevanlinna functions in several variables, their admissible measures and the corresponding operators. It can be adjusted to the background and the interests of the candidate.*

Possible questions include:

- Identify the representing measures for subclasses of Herglotz-Nevanlinna functions, e.g., rational ones.
- Find more necessary conditions for a measure to be an admissible measure.
- Identify the admissible measures that give rise to those functions which also have a representation with a resolvent och find an explicit relation between the measure and the corresponding operator.
- Find extremal measures.

References

[AMY] J. Agler, J. E. McCarthy, and N. J. Young, *Operator monotone functions and Löwner functions of several variables*, Ann. of Math. (2) **176** (2012), no. 3, 1783–1826.

[ATuY] J. Agler, R. Tully-Doyle, and N. J. Young, *Nevanlinna representations in several variables*, J. Funct. Anal. **270** (2016), no. 8, 3000–3046.

[T] G. Teschl, *Gerald Mathematical methods in quantum mechanics. With applications to Schrˆdinger operators*, Second edition. Graduate Studies in Mathematics, 157, American Mathematical Society, Providence, RI, 2014. xiv+358 pp.

[LN1] A. Luger and M. Nedic: *Herglotz-Nevanlinna functions in several variables*, J. Math. Anal. Appl., Volume **472**, Issue 1, 1189-1219