The combinatorics of addition: number theory, harmonic analysis and probability

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Questions about set addition have been studied for centuries, particularly in number theory. We know, for instance, that any positive integer can be written as the sum of four squares, and that every odd integer greater than 5 can be written as a sum of three primes. However, even about these specific sets of integers — squares and primes — there are many things we do not know. A relatively recent trend ('additive combinatorics') is to try to understand set addition much more generally, and in particular to understand under what conditions on a set A of integers we can deduce interesting additive information. In fact, it is often fruitful to consider subsets not just of the integers, but of abelian groups more generally.

The overall purpose of this project is to improve our understanding of additive structures in subsets A of abelian groups, and their sumsets $A + A$, $A + A + A$ etc. In particular, we aim to make use of and further develop recent advances in the area that at their core use harmonic analysis and random sampling to uncover additive structure.

One departure point for study will be the fact, proved in the last few years [1, 2, 6], that the primes contain infinitely many three-term arithmetic progressions, and so in fact do quite sparse subsets of the primes, simply by virtue of there being quite a lot of primes in any interval $\{1, 2, \ldots, N\}$ (namely about $N/\log N$). Indeed, we now know that *any* set that has this many elements in such an interval must contain a lot of additive structure. The techniques use (among other things) results in harmonic analysis that go under the name almost-periodicity, with roots in [4], as well as random sampling arguments that are naturally viewed probabilistically. Part of this project will involve investigating whether some of these techniques can be unified and/or generalised to other settings, as well as to investigate their limitations. Other relevant techniques [3, 5] will also be an early point of investigation.

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References

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