Powers of general linear forms

Supervisor: Samuel Lundqvist, samuel@math.su.se

The project aims at gaining a deeper insight of the structure of ideals generated by powers of general linear forms. These ideals may be regarded as geometric objects, so called fat-points schemes, making it possible to study the problem from two different perspectives. The project is supported by a grant from the Swedish Research Council.

Detailed project description

Consider the monomial complete intersection algebra

$$R = \mathbb{C}[x_1, \ldots, x_n]/(x_1^{d_1}, \ldots, x_n^{d_n}).$$

By assigning each variable degree one, this algebra becomes graded; $R = \oplus_{i\geq 0} R_i$, and each graded component is a vector space over $\mathbb{C}$. Let us pick a general linear form ℓ in R. For each i and each j the element ℓ^j induces a map $R_i \to R_{i+j}, a \mapsto \ell^j a$. It is known that for all i and j, this map is either surjective or injective, which by definition means that $R = \mathbb{C}[x_1, \ldots, x_n]/(x_1^{d_1}, \ldots, x_n^{d_n})$ has the *strong Lefschetz property.* This result is due to Stanley [4]. We have in fact that R is isomorphic to $R = \mathbb{C}[x_1, \ldots, x_n]/(\ell_1^{d_1}, \ldots, \ell_n^{d_n})$, where each ℓ_i is a general linear form (make a linear change of coordinates). Moreover, the Hilbert series of R is known – it equals

$$\prod_{i=1}^{n}(1-t)^{d_i}/(1+t)^n = \prod_{i=1}^{n}(1+t+t^2+\cdots+t^{d_i-1}).$$

These facts gives the following result: The Hilbert series of the algebra $\mathbb{C}[x_1, \ldots, x_n]/(\ell_1^{d_1}, \ldots, \ell_{n+1}^{d_{n+1}})$ equals

$$\left[\prod_{i=1}^{n+1}(1-t)^{d_i}/(1+t)^n\right],$$

where the brackets means truncate at the first non-positive entry.

Given the above, one could guess that the Hilbert series of the algebra $\mathbb{C}[x_1, \ldots, x_n]/(\ell_1^{d_1}, \ldots, \ell_{n+2}^{d_{n+2}})$ would equal $[\prod_{i=1}^{n+2}(1-t)^{d_i}/(1+t)^n]$, but this is far from correct, and for the equigenerated case $d_1 = \cdots = d_{n+2}$ only true for some special cases [1]. In particular this means that for these cases $\mathbb{C}[x_1, \ldots, x_n]/(\ell_1^{d_1}, \ldots, \ell_{n+1}^{d_{n+1}})$ fails even the weak Lefschetz property.

For $m \geq n+4$ the Hilbert series of $\mathbb{C}[x_1, \ldots, x_n]/(\ell_1^{d_1}, \ldots, \ell_m^{d_n})$ does however equal $[\prod_{i=1}^{m}(1-t)^{d_i}/(1+t)^n]$ in many cases, and Iarrobino [3] has conjectured that equality holds in the equigenerated case (except for some special cases).

The project aims at gaining a deeper knowledge of the Hilbert series of the algebra $\mathbb{C}[x_1, \ldots, x_n]/(\ell_1^{d_1}, \ldots, \ell_m^{d_m})$ for $m \geq n+2$, and making progress on the longstanding and important Fröberg conjecture [2], on Interpolation theory, and in particular, on Iarrobino's conjecture.

There is reason to believe that computer experiments, the theory of fat points schemes, and Macaulay's inverse system would be key concepts for doing progress on the problems. The applicant is expected to have a solid background in algebra, at least a basic knowledge of programming, and a willingness to perform computer experiments.

For questions about the project, please send me an email.

References

1. M. Boij and S. Lundqvist, A classification of the weak Lefschetz property for almost complete intersections generated by uniform powers of general linear forms. Accepted for publication in Alg. Number Th. 17–1, 111–126 (2023).

2. R. Fröberg, An inequality for Hilbert series, Math. Scand. 56, 117–144 (1985).

3. A. Iarrobino, Inverse system of a symbolic power III: thin algebras and fat points, Compos. Math. (3) 108,319–356 (1997).

4. R. Stanley, Weyl groups, the hard Lefschetz theorem, and the Sperner property, SIAM J. Algebraic Discrete Methods, 1(2):168–184 (1980).