RANDOM PROCESSES WITH REINFORCEMENT

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Random processes with reinforcement are stochastic processes that exhibit long-term coordination, in the sense that the entire trajectory of the process contributes to its eventual fate. Polya's urn model, introduced around a century ago, is the standard example of a process of this type. In this model, a finite number of red and blue balls are initially placed in an urn. In each round of the process, a ball is drawn uniformly at random from the urn, and replaced together with a ball of the same colour. The drawn colour is thus reinforced, in that balls of the same colour are more likely to be drawn in future steps. The effect of the reinforcement declines over time, resulting in the early steps of the process having a lasting effect over its continued evolution, and the limiting proportion of red balls in Pólya's urn being random.

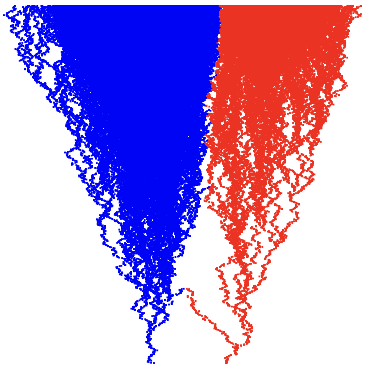

FIGURE 1. The formation of a linear interface between red and blue in annihilating branching Brownian motion. The slope of the interface is random, as a consequence of the reinforcement mechanism built into the process.

Random processes with reinforcement arise in various context, including economics and evolutionary biology. Branching processes and other martingale methods are central techniques in their study; see e.g. [4]. *Branching Brownian motion* and *branching random walks* are examples of reinforcement processes with a spatial component. In these models particles move around according to Brownian motion or random walks, and split into independent particles at unit rate. Interesting phenomena have recently been discovered in *annihilating* versions of such processes, where particles are of one out of two types (red and blue, say), and where particles of different type annihilate upon contact. The annihilating behaviour turns the growth process into a model for competing growth.

One key observation in the presence of annihilation is the impossibility of eventual coexistence of the two types in a compact setting [3, 2], and the possibility of coexistence in an unbounded setting [1]. However, much remains unknown regarding these processes, and several open problems and conjectures are stated in the mentioned papers. For instance, in the unbounded setting,

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on the event of coexistence, the interface separating red and blue particles is known to evolve linearly. The slope of the interface is random, as a consequence of the reinforcement mechanism (see figure). However, finer details regarding the slope of the interface remain unknown, such as its fluctuations around its random limit.

Animated introductions to some of the results described above are available here:
Annihilating branching Brownian motion
Multi-colour competition with reinforcement

References

[1] Daniel Ahlberg, Omer Angel, and Brett Kolesnik. Annihilating branching Brownian motion. *International Mathematical Research Notices*, to appear.

[2] Daniel Ahlberg and Carolina Fransson. Multi-colour competition with reinforcement. *Ann. Inst. Henri Poincaré Probab. Stat.*, to appear.

[3] Daniel Ahlberg, Simon Griffiths, Svante Janson, and Robert Morris. Competition in growth and urns. *Random Structures Algorithms*, 54(2):211–227, 2019.

[4] Robin Pemantle. A survey of random processes with reinforcement. *Probab. Surv.*, 4:1–79, 2007.